

COSMOLOGICAL CONSTANT AND GRAVITATIONAL REPULSION EFFECT: 1. Homogeneous models with radiation

Nguyen Hong Chuong^{*†}

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA

Nguyen Van Hoang

Centre for Cosmic Physics and Remote Sensing , 20000 Hochiminh City, Vietnam

Abstract

Within the framework of the minimum quadratic Poincare gauge theory of gravity in the Riemann-Cartan spacetime we study the influence of gravitational vacuum energy density (a cosmological constant) on the dynamics of various gravitating systems. It is shown that the inclusion of the cosmological term can lead to gravitational repulsion. For some simple cases of spatially homogeneous cosmological models with radiation we obtain non-singular solutions in form of elementary functions and elliptic integrals.

PACS number(s): 04.50.+h, 98.80.+k

I. INTRODUCTION

In the Poincare gauge theory of gravity (PGT) the gravity is described by the field of tetrads h^i_μ and the Ricci rotation coefficients A^{ik}_ν [1-4]. Due to the dynamical independence of these gauge fields the spacetime has both the curvature $F^{ik}_{\mu\nu}$ and the torsion $S^i_{\mu\nu}$. Usually, one connects the torsion with the spin angular momentum of matter. In the simplest PGT - the Einstein-Cartan theory (ECT) [1-2] - by virtue of the algebraic dependence between the spin angular momentum and the torsion, we recover General Relativity (GR) for the spinless gravitating systems.

In the PGT, the choice of gravitational Lagrangian is very important. The principle of local gauge invariance itself gives only the form of the gauge fields and their strengths but does not allow us to determine the explicit form of the Lagrangian for the gauge field. As was shown in [4,5] the simplest generalization of the ECT, which satisfies the restrictions following from the simultaneous consideration of the quantization problem (theory without "ghosts" and "tachyons") [6], the Birkhoff's theorem [7], and the problem of avoiding the metric singularity in the homogeneous isotropic cosmological models [8, 9], is the minimum quadratic gauge theory of gravity (MQGT) with the gravitational Lagrangian:

$$L_g = h(f_0 F + \alpha F^2), \quad F_{\mu\nu} = F^\lambda_{\mu\lambda\nu}, \quad F = F^\mu_\mu, \quad (1.1)$$

where $h = \det(h^i_\mu)$, F is the scalar curvature of the Riemann-Cartan spacetime, $f_0 = (16\pi G)^{-1}$, (G is the Newton's gravitational constant), and $\alpha < 0$ is a dimensionless coefficient.

In [5] the basic field equations of MQGT were derived and in [10-14] some of their physical consequences were investigated. In this paper we will study the influence of the cosmological term on the dynamics of some spatially homogeneous gravitating systems. This work is a direct continuation of [10] and we will use all notations in [10], unless otherwise stated.

II. COSMOLOGICAL TERM AND GRAVITATIONAL REPULSION EFFECT.

According to the modern viewpoint, the cosmological term is connected with the energy density of the gravitating vacuum. Taking into account the cosmological term χ in the Lagrangian (1.1), two field equations of MQGT are obtained by independent variation of (1.1) with respect to h^i_μ and A^{ik}_ν :

$$2(f_0 + 2\alpha F)F^\mu_i - (f_0 + \alpha F)F h^\mu_i + \chi h^\mu_i = -t^\mu_i, \quad (2.1)$$

$$2\nabla_\nu [(f_0 + 2\alpha F)h^{[\mu}_i h_k]^{\nu]} = -J_{ik}{}^\mu. \quad (2.2)$$

In this work we restrict ourselves to consider ones the so-called conformally-invariant gravitating systems of spinless matter with trace-free energy-momentum tensor:

$$J_{ik}{}^\nu = 0 \quad \text{and} \quad t_\mu^\mu = 0. \quad (2.3)$$

It then follows from (2.1) that $F = \frac{2\chi}{f_0} = \text{const}$, and consequently, from (2.2) we find

$$S^\lambda{}_{\mu\nu} = 0. \quad (2.4)$$

After some simple manipulation from (2.1) we obtain the MQGT equation for the considered systems in the form of the Einstein's equation of GR with the effective energy-momentum tensor as:

$$G^{\lambda\mu}(\{\}) = -\frac{1}{2f_0} \left(t_{eff}^{\lambda\mu} + \chi g^{\lambda\mu} \right) = -\frac{1}{2f_0} \left(\frac{t^{\lambda\mu}}{1 - 4\beta\chi} + \chi g^{\lambda\mu} \right), \quad (2.5)$$

where $G^{\lambda\mu}(\{\})$ is the ordinary Einstein tensor, $\beta = -\frac{\alpha}{f_0^2} > 0$, and $\chi \neq \frac{1}{4\beta}$. It is obvious that if $\chi = 0$ we recover ordinary Einstein's equation (cf. refs [5,13]). If $\chi < \frac{1}{4\beta}$ the solutions of eq. (2.5) cannot provide any qualitatively new properties in comparison with the corresponding solutions of GR.

However, under the condition

$$\chi > \frac{1}{4\beta} \quad (2.6)$$

the effective energy-momentum tensor $t_{eff}^{\lambda\mu}$ in (2.5) is negative, that leads to the gravitational repulsion effect. This effect is induced by a large cosmological constant (high gravitating vacuum energy density) and by the square curvature term in gravitational Lagrangian (1.1), although the torsion is vanishing. Due to this repulsion effect, the Penrose-Hawking energy conditions [15] might be broken and we can find a series of non-singular solutions, which are of certain interest for astrophysical and cosmological applications. Note that in the very early stage of the cosmological evolution the vacuum energy density χ was, in fact, very high.

In the following sections on the basis of (2.5) we will study the gravitational repulsion effect in some simple spatially homogeneous cosmological models with ultrarelativistic perfect

fluid (isotropic radiation) with the following energy-momentum tensor and equation of state:

$$t^{\lambda\mu} = (\rho + p)u^\lambda u^\mu - pg^{\lambda\mu} \quad (2.7)$$

$$p = \frac{\rho}{3} \quad (2.8)$$

III. REGULAR FRIEDMANN-ROBERTSON-WALKER MODELS WITH RADIATION.

We consider the homogeneous isotropic cosmological models with the Friedmann-Robertson-Walker metric:

$$g_{\mu\nu} = \text{diag}\left(1, -\frac{R^2(t)}{1 - kr^2}, -R^2(t)r^2, -R^2(t)r^2 \sin^2\theta\right), \quad (3.1)$$

where $R(t)$ is scale factor, and $k = -1, 0, +1$ for open, flat and closed models correspondingly. Then, the 00 - component of (2.5) takes the form:

$$\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} = \frac{1}{6f_0} \left(\frac{\rho}{1 - 4\beta\chi} + \chi \right), \quad (3.2)$$

where a dot denotes differentiation with respect to time t . The “energy-momentum conservation law” has the standard form

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0. \quad (3.3)$$

It follows from Eqs. (3.3) and (2.8) that

$$\rho R^4 = C_0^2 = \text{const} > 0. \quad (3.4)$$

Substituting (3.4) into (3.2) we obtain after simple manipulations the differential equation for scale factor $R(t)$:

$$\dot{R}^2 = \frac{\chi}{6f_0} R^2 - k - \frac{C_0^2}{6f_0(4\beta\chi - 1)R^2}. \quad (3.5)$$

In the case considered, when $\chi > \frac{1}{4\beta}$ it is easy to find the following regular solutions

$$R(t) = \left\{ \frac{3kf_0}{\chi} + \left(\frac{9k^2f_0^2}{\chi^2} + \frac{C_0^2}{\chi(4\beta\chi - 1)} \right)^{\frac{1}{2}} \cosh\left[\left(\frac{2\chi}{3f_0}\right)^{\frac{1}{2}}(t - t_0)\right] \right\}^{\frac{1}{2}}. \quad (3.6)$$

In fact, the scale factor (3.6) has the positive minimum value

$$R_{min} = \frac{3kf_0}{\chi} + \left(\frac{9k^2f_0^2}{\chi^2} + \frac{C_0^2}{\chi(4\beta\chi - 1)} \right)^{\frac{1}{2}} > 0, \quad (3.7)$$

not only for closed and flat models ($k = 1, 0$), but also for open models, when $k = -1$ by virtue $4\beta\chi - 1 > 0$. Consequently, the energy density is limited $\rho \leq \rho_{max} = C_0^2/R_{min}^4$.

On the basis of the obtained solution we can construct the regular homogeneous isotropic (inflationary) cosmological models and the regular models of superdense isotropic radiation systems in the framework of MQGT (cf. refs. [16, 17]).

IV. REGULAR BIANCHI TYPE-I MODELS WITH RADIATION

We now study the homogeneous Bianchi type-I radiation models with the metric given in form:

$$g_{\mu\nu} = \text{diag} \left(1, -r_1^2(t), -r_2^2(t), -r_3^2(t) \right). \quad (4.1)$$

It follows from kk -components of eq. (2.5) ($k = 1, 2, 3$) the following relationships:

$$r_k = r \exp \left(s_k \int \frac{dt}{r^3} \right), \quad (4.2)$$

where $r = (r_1 r_2 r_3)^{1/3}$; $\sum_{k=1}^3 s_k = 0$. Then, after some transformations the 00-component of eq. (2.5) and "energy-momentum conservation law" can be written as:

$$\frac{\dot{r}^2}{r^2} = \frac{1}{6f_0} \left(\frac{\rho}{1 - 4\beta\chi} + \chi \right) + \frac{I^2}{r^6}, \quad (4.3)$$

$$\rho r^4 = C_I^2 = \text{const} > 0, \quad (4.4)$$

where $I^2 = \frac{1}{6} \sum_{k=1}^3 s_k^2$, is a measure of the anisotropy. By using (4.4) it is easy to find the exact solution of (4.3) in the following analytic form

$$t - t_0 = \left(\frac{3f_0}{2\chi} \right)^{\frac{1}{2}} \int \left(\frac{x}{P_I(x)} \right)^{\frac{1}{2}} dx, \quad (4.5)$$

$$P_I(x) = x^3 - \frac{C_I^2}{\chi(4\beta\chi - 1)} x + \frac{6f_0 I^2}{\chi}, \quad (4.6)$$

where $x = r^2 > 0$. It is clear that the integral (4.5) makes sense only if $P_I(x) > 0$. Note that for all $x \geq x_1$, where x_1 is the greatest real root of the cubic polynomial $P_I(x)$, $P_I(x) \geq 0$ holds. If $x_1 \leq 0$, $P_I(x) \geq 0$ for all $x \geq 0$, and therefore, the solution (4.5) is singular because the function $x(t) = 0$ at a finite time $t = t_0$. However, if $x_1 > 0$ the solution (4.5) is regular: $x \geq x_1 > 0$ for all $t \in (-\infty, +\infty)$. Thus, in order to get non-singular solution from (4.5) we have to find conditions for the existence of positive real roots of the cubic polynomial $P_I(x)$. Detailed analysis shows, that under the conditions (2.6) and

$$I^2 \leq A_I(C_I, \beta, \chi) = \frac{C_I^3}{\sqrt{\chi} [3(4\beta\chi - 1)]^{3/2}} \quad (4.7)$$

the cubic polynomial $P_I(x)$ possesses three real roots $x_1 \geq x_2 > 0 > x_3$. Then, for $x \geq x_1$ the solution (4.5) is regular and can be reduced to the elliptic integrals Π , \mathcal{F} (see Ref. [10])

$$t - t_0 = \left(\frac{6f_0}{x_1(x_2 - x_3)\kappa} \right)^{\frac{1}{2}} \left[(x_1 - x_2) \Pi\left(\phi, \frac{x_1 - x_3}{x_2 - x_3}, \kappa\right) + x_2 \mathcal{F}(\phi, \kappa) \right], \quad (4.8)$$

where

$$\phi = \arcsin \left[\left(\frac{(x_2 - x_3)(x - x_1)}{(x_1 - x_3)(x - x_2)} \right)^{\frac{1}{2}} \right]; \quad \kappa = \left(\frac{x_2(x_1 - x_3)}{x_1(x_2 - x_3)} \right)^{\frac{1}{2}}. \quad (4.9)$$

The minimum value of the universe "volume" is positive $V_{min} = r_{min}^3 = x_1^{3/2} > 0$, where

$$x_1 = \left(\frac{4C_I^2}{3\chi(4\beta\chi - 1)} \right)^{\frac{1}{2}} \cos \left[\frac{1}{3} \arccos \left(9\sqrt{3}f_0 I^2 \sqrt{\chi} (4\beta\chi - 1)^{3/2} C_I^{-3} \right) \right], \quad (4.10)$$

and the maximum value of energy density ρ is limited: $\rho_{max} = C_I^2/x_1^2$.

Thus, under the conditions (2.6) and (4.7) the vacuum (repulsion) effect dominates over ordinary gravitational contraction effect and anisotropic effect (that is sufficiently weak), so permits avoiding the singularity in the Bianchi type-I cosmological models with radiation.

V. REGULAR BIANCHI TYPE-V MODELS WITH RADIATION

In this section we consider the Bianchi type-V models that are a direct generalization of open Friedmann-Robertson-Walker models. Taking (x_1, x_2, x_3) as local coordinates we

can write the interval of Bianchi type-V models in the diagonal form [18, 19]

$$ds^2 = dt^2 - r_1^2(t)dx_1^2 - e^{2x_1} \left(r_2^2(t)dx_2^2 + r_3^2(t)dx_3^2 \right). \quad (5.1)$$

For the models considered we can find the following non-vanishing components of the Einstein's tensor

$$\begin{aligned} G_0^0 &= \frac{1}{2} \sum_{k=1}^3 \frac{\dot{r}_k^2}{r_k^2} - \frac{9}{2} \frac{\dot{r}^2}{r^2} + \frac{3}{r_1^2}, \\ G_1^0 &= 2 \frac{\dot{r}_1}{r_1} - \left(\frac{\dot{r}_2}{r_2} + \frac{\dot{r}_3}{r_3} \right), \\ G_i^i &= \left(\frac{\dot{r}_i}{r_i} \right)^{\cdot} - 3 \left(\frac{\dot{r}}{r} \right)^{\cdot} + 3 \frac{\dot{r} \dot{r}_i}{r r_i} - \frac{1}{2} \sum_{k=1}^3 \frac{\dot{r}_k^2}{r_k^2} - \frac{9}{2} \frac{\dot{r}^2}{r^2} + \frac{1}{r_1^2} \quad (no \text{ sum}), \end{aligned} \quad (5.2)$$

where $r = (r_1 r_2 r_3)^{1/3}$. It follows from $G_1^0 = 0$ that

$$\frac{\dot{r}_1}{r_1} = \frac{1}{2} \left(\frac{\dot{r}_2}{r_2} + \frac{\dot{r}_3}{r_3} \right) = \frac{\dot{r}}{r}, \quad (5.3)$$

and, consequently $r_1 = r$ (by rescaling). From (2.5) we get $G_1^1 = G_2^2 = G_3^3$ and by virtue of (5.3) we can find

$$r_2 = r \exp \left(s \int \frac{dt}{r^3} \right); \quad r_3 = r \exp \left(-s \int \frac{dt}{r^3} \right), \quad (5.4)$$

where s is an integration constant and has the sense of a measure of the anisotropy. Then, 00-component of (2.5) can be reduced to

$$\frac{\dot{r}^2}{r^2} = \frac{1}{6f_0} \left(\frac{\rho}{1-4\beta\chi} + \chi \right) + \frac{s^2}{3r^6} + \frac{1}{r^2}, \quad (5.5)$$

The "energy-momentum conservation law" has the ordinary form

$$\rho r^4 = C_V^2 = const > 0. \quad (5.6)$$

By analogy with the foregoing section we obtain the exact solution of (5.5) in an analytic form

$$t - t_0 = \left(\frac{3f_0}{2\chi} \right)^{\frac{1}{2}} \int \left(\frac{x}{P_V(x)} \right)^{\frac{1}{2}} dx, \quad (5.7)$$

$$P_V(x) = x^3 + \frac{6f_0}{\chi} x^2 - \frac{C_V^2}{\chi(4\beta\chi - 1)} x + \frac{2f_0 s^2}{\chi}, \quad (5.8)$$

where $x = r^2 > 0$.

By analogy with section 4, together with condition (2.6) we can find the similar to (4.7) condition for the existence of positive real root of the cubic polynomial $P_V(x)$ as

$$s^2 \leq A_V(C_V, \beta, \chi) = x_+ \left(\frac{C_V^2}{3f_0(4\beta\chi - 1)} - x_+ \right), \quad (5.9)$$

where

$$x_+ = \left(\frac{4f_0^2}{\chi^2} + \frac{C_V^2}{3\chi(4\beta\chi - 1)} \right)^{\frac{1}{2}} - \frac{2f_0}{\chi}. \quad (5.10)$$

It is easy to show that $A_V(C_V, \beta, \chi) > 0$, and consequently, the condition (5.9) makes sense. Note that condition (5.9) gives stronger a restriction on the anisotropy than (4.7). In the case considered, the vacuum (repulsion) effect dominates over the ordinary gravitational attraction effect, the anisotropic effect, and the effect of negative curvature.

Thus, under the conditions (2.6) (large cosmological constant) and (5.9) (small anisotropy) the solution (5.7) is regular and has the same form as (4.8). Then, we can construct the non-singular Bianchi type-V radiation cosmological models with the limiting energy density $\rho \leq \rho_{max} < \infty$, and nonzero metric functions $r_i(t) > 0$ at any time.

VI. CONCLUSIONS

In this paper we have shown that the gravitating vacuum effect can be of importance for avoiding the singularity in some homogeneous cosmological models with the isotropic radiation. We obtain regular solutions for Friedmann-Robertson-Walker, Bianchi type-I, and Bianchi type-V models. These solutions have been expressed in terms of elementary functions and elliptic integrals. Note that the gravitational repulsion effect can play an important role in the case of conformally-invariant systems such as: massless conformally-invariant scalar field, electromagnetic field... On the basis of eq. (2.5) following [20] we can find exact non-singular solutions for various cosmological models with radiation, scalar, and electromagnetic fields, as well as for spherically-symmetric conformally-invariant systems.

ACKNOWLEDGMENTS

The authors are grateful to Profs. A. V. Minkevich and F. I. Fedorov for their guidance during their study in Minsk. N. H. Chuong would like to thank Profs. S. Bazanski, and W. Kopczynski for warm hospitality in Warsaw, where the first draft of the paper was presented. He would especially like to thank Prof. A. Zichichi and the World Laboratory for providing the fellowship and Prof. A. Ashtekar for kind attention during his stay in Syracuse. This work was supported in part by research funds provided by Syracuse University.

REFERENCES

- * On leave of absence from Institute of Theoretical Physics, P. O. Box 429 Boho, 10000 Hanoi, Vietnam.
- † Electronic mail address: chuong@suhep.phy.syr.edu
- [1] F.W. Hehl, P. von der Heyde, G. Kerlick, and M. Nester, *Rev. Mod. Phys.*, **48**, 393 (1976)
- [2] D. Ivanenko, and G. Sardanasvily, *Phys. Rep.*, **94**, 1 (1983)
- [3] P. Baekler, F.W. Hehl, and E.W. Mielke, in *Proceedings of the 4th M. Grossmann Meeting on General Relativity* Ed. R. Ruffini, (Elsevier Science Publ., Amsterdam. 1986)
- [4] F.I. Fedorov, V.I. Kudin, and A.V. Minkevich, *Acta Phys. Polon.*, **B15**, 107 (1984)
- [5] A.V. Minkevich, *Izv. Akad. Nauk BSSR: Fiz. Mat.*, No5, 100 (1986)
- [6] E. Sezgin, and P. van Nieuwenhuizen, *Phys. Rev.*, **D21**, 3269 (1980)
- [7] R. Rauch, H.T. and Nieh, *Phys. Rev.*, **D24**, 2029 (1981)
- [8] A.V. Minkevich, *Phys. Lett.*, **A80**, 232 (1980)
- [9] H. Goenner, and F. Müller-Hoissen, *Class. Quantum Grav.*, **1**, 651 (1984)
- [10] Nguyen Hong Chuong, *Phys. Lett.*, **A142**, 326 (1989)
- [11] Nguyen Hong Chuong, *Gen. Relat. Gravit.*, **22**, 1229 (1990)
- [12] A.V. Minkevich, Nguyen Hong Chuong, and F.I. Fedorov, *Izv. Akad. Nauk BSSR: Fiz. Mat.*, No1, 98 (1987)
- [13] Nguyen Hong Chuong, C.Sc.Thesis, Byelorussian State University, Minsk, 1986
- [14] Nguyen Hong Chuong, and Nguyen Van Hoang, Syracuse University preprint No. SU-GP-93/3-4, 1993; submitted to *J. Math. Phys.*

- [15] S. Hawking, and G.F.R. Ellis, The Large Scale Structure of Space-time, (Cambridge University Press, Cambridge, 1973)
- [16] A.V. Minkevich, *Dokl. Akad. Nauk BSSR*, **30**, 311 (1986)
- [17] A.V. Minkevich, and Nguyen Hong Chuong, *Izv. Akad. Nauk BSSR: Fiz. Mat.*, No5, 103 (1989)
- [18] Nguyen Van Hoang, C.Sc. Thesis, Byelorussian State University, Minsk, 1990
- [19] D. Lorentz, *Gen. Relat. Gravit.*, **13**, 795 (1981)
- [20] D. Kramer, H. Stephani, M.A.H. MacCallum, and E. Herlt, Exact Solutions of Einstein's Field Equations, (VEB, Berlin, 1980)